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Random Motions at Finite Speed in Higher Dimensions

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Abstract We present a general method of studying the transport process $\mathbf{X}(t)$, $t \geq 0$, in the Euclidean space $\mathbb{R}^m$, $m \geq 2$, based on the analysis of the integral transforms of its distributions. We show that the joint characteristic functions of $\mathbf{X}(t)$ are connected with each other by a convolution-type recurrent relation. This enables us to prove that the characteristic function (Fourier transform) of $\mathbf{X}(t)$ in *any* dimension $m \geq 2$ satisfies a convolution-type Volterra integral equation of second kind. We give its solution and obtain the characteristic function of $\mathbf{X}(t)$ in terms of the multiple convolutions of the kernel of the equation with itself. An explicit form of the Laplace transform of the characteristic function in any dimension is given. The complete solution of the problem of finding the initial conditions for the governing partial differential equations, is given.

We also show that, under the standard Kac condition on the speed of the motion and on the intensity of the switching Poisson process, the transition density of the isotropic transport process converges to the transition density of the m-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient depending on the dimension m.

We give the conditional characteristic functions of the isotropic transport process in terms of the inverse Laplace transform of the powers of the Gauss hypergeometric function. Some important models of the isotropic transport processes in lower dimensions are considered and some known results are derived as the particular cases of our general model by means of the method developed.

Keywords Random motion · Finite speed · Transport process · Random evolution · Characteristic function · Bessel function · Convolution · Volterra integral equation · Fourier transform · Laplace transform · Multidimensional Brownian motion · Initial conditions

1 Introduction

The diffusion processes with finite speed of propagation are highly appropriate models for describing various real phenomena in statistical physics, hydrodynamics, biology and other

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fields. The diffusion with finite speed of propagation is generated by random motions of a particle that moves with finite speed in some phase space. Such a type of random motion is referred to as the transport process, random flight or, in a more general sense, random evolution. While the finiteness of the velocity is the basic feature of such motions, the models differ with respect to the way of choosing the new directions (the scattering function) and the dimension of the phase space.

The one-dimensional models of random evolution (Goldstein-Kac telegraph process and its numerous generalizations) have been intensively studied by many researchers and a great deal of results were obtained. As to the multidimensional counterparts of this process is concerned, many efforts were made to extend this model of random motion to higher-dimensional spaces $\mathbb{R}^m$, $m \geq 2$. The main difficulty in doing so steams from the fact that, in contract to the one-dimensional case, there exists a continuum of possible directions in any other dimension $m \geq 2$. For this reason the idea of discretization of the continuous spectrum of directions came to dominate in this field of research for many years. Now there exists a great variety of the works devoted to random motions with a finite number of directions (for the most general model of a cyclic random motion in arbitrary dimension see the recent paper by Lachal [10] and the bibliography therein).

While the motions with a finite number of directions can be of interest in some specific models, the evolutions with a continuum of directions are, undoubtedly, much more natural and practically useful. That's why the problem of describing the transport processes with a continuous spectrum of directions in higher dimensions $m \geq 2$ is of a special importance.

In the study of such processes the most desirable goal is, undoubtedly, their explicit distributions in those cases (very few indeed), when such distributions can be obtained. The transition density of a two-dimensional random motion has been derived by Stadje [15] by means of recursive arguments. The similar result for a planar isotropic transport process was given by Masoliver *et al.* [11] by means of recursive arguments and Fourier-Laplace transforms. The same result has been derived by Kolesnik and Orsingher [7] by using the characteristic functions techniques and reobtained in Kolesnik [6] by means of some specific properties of the wave propagation in the plane.

A three-dimensional transport process with an arbitrary bounded scattering function was studied by Tobulinsky [17, Chap. 2, pp. 35–60], and the transition density of the process was given in terms of the resolvent of an integral operator. The similar three-dimensional random motion with the uniform choice of directions was examined by Stadje [16] and the transition density of the process was given in the form of a fairly complicated integral with variable limits, which seemingly cannot be explicitly evaluated in terms of elementary functions.

A four-dimensional isotropic transport process has recently been studied by Kolesnik [3] and by Orsingher and De Gregorio [12] and the transition density of the motion has been obtained in a fairly simple analytical form by means of the characteristic functions techniques.

These works can be considered as the successive steps toward the most desirable goal, namely, constructing a general theory of the distributions for transport processes in the Euclidean spaces $\mathbb{R}^m$, $m \geq 2$, and, in a more general setting, on the manifolds. However, the behaviour of such random motions in diverse Euclidean spaces is a genuine enigma. The effect of dimensionality is the core of this puzzle. It's hard to explain the considerable distinguishes in the forms of the transition densities of transport processes in different dimensions. While in the two- and four-dimensional cases the transition densities have fairly simple analytical forms, the three-dimensional transition density has a very complicated integral form and, apparently, cannot be expressed in terms of elementary functions.

The principal aim of this paper is to link together all the known particular models of the random motions at finite speed in the Euclidean spaces $\mathbb{R}^m$, $m \geq 2$, and give the most general formulae applicable in *any* dimension. Our goal is to present an unified approach to the problem of describing the transport processes based on the analysis of the integral transforms of their distributions. The main tools of our research are the characteristic functions (Fourier transforms) of the distributions involved as well as their Laplace transforms.

The paper is organized as follows. In Sect. 2 we consider a m-dimensional isotropic transport process $\mathbf{X}(t)$, $t \geq 0$, performed by a particle that moves with constant finite speed c in the Euclidean space $\mathbb{R}^m$, $m \geq 2$, and whose motion is controlled by a homogeneous Poisson process of rate $\lambda > 0$. At the Poissonian instants the particle takes on the new direction with uniform law on the unit sphere. The main result of Sect. 2 states that the joint (as well as conditional) characteristic functions (with respect to the number of the Poissonian events that have occurred by time t) are connected with each other by the convolution-type recurrent relations. This enables us to derive some very important relations for the Laplace transforms of the joint and conditional characteristic functions.

In Sect. 3 we derive a formula for the conditional characteristic functions of $\mathbf{X}(t)$, $t \geq 0$, in terms of the inverse Laplace transform of the powers of the Gauss hypergeometric function. We obtain, as the particular cases of our general model, the explicit forms of the conditional characteristic functions for the two- and four-dimensional isotropic transport processes previously obtained by other methods in the works mentioned above. The three-dimensional case is also analyzed by our techniques.

In Sect. 4 we obtain a very important result stating that, in *any* dimension $m \geq 2$, the characteristic function $H(t)$ of the process $\mathbf{X}(t)$ satisfies a convolution-type Volterra integral equation of second kind with continuous kernel. We solve this equation and give its solution (which is unique in the class of continuous functions) in terms of the multiple convolutions of the kernel of the equation with itself. An explicit form of the Laplace transform of $H(t)$ is presented. We also give the complete solution of the problem of finding the initial conditions for the partial differential equations governing the isotropic transport processes.

In Sect. 5 we study the limiting behaviour of $\mathbf{X}(t)$. We prove that, under the standard Kac condition on the speed c of the motion and on the rate λ of the governing Poisson process, the transition density of $\mathbf{X}(t)$ converges to the transition density of the m-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient depending on the dimension m.

In Sect. 6 we consider a non-symmetrical random motion and give the non-symmetrical counterparts of the main formulae obtained in the previous sections for the isotropic transport processes. Surprisingly, the majority of the results valid for symmetrical random motions can easily be extended for the non-symmetrical case.

In Appendix we prove two auxiliary lemmas which have been used in our analysis.

The main results of this paper were announced in Kolesnik [5].

2 Structure of Distribution and Recurrent Relation

We consider a particle starting its motion from the origin $\mathbf{0} = (0, \dots, 0)$ of the space $\mathbb{R}^m$, $m \geq 2$, at time $t = 0$. The particle moves with constant, finite speed c (note that c is treated as the constant norm of the velocity). The initial direction is a random m-dimensional vector with uniform distribution (Lebesgue probability measure) on the unit sphere

$$S_1^m = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : x_1^2 + \cdots + x_m^2 = 1 \right\}.$$

We note at once that here and thereafter the upper index m means the dimension of the space in which the sphere S_1^m is considered, not its own dimension which, clearly, is $m - 1$.

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on S_1^m, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))$ be the position of the particle at an arbitrary time $t > 0$. In this section we concentrate our attention on the conditional distributions

$$\begin{aligned} &Pr\{\mathbf{X}(t) \in d\mathbf{x} \mid N(t) = n\} \\ &\quad = Pr\{X_1(t) \in dx_1, \dots, X_m(t) \in dx_m \mid N(t) = n\}, \quad n \geq 1 \end{aligned}$$

where $N(t)$ is the number of Poisson events that have occurred in the interval $(0, t)$ and $d\mathbf{x}$ is the infinitesimal element in the space $\mathbb{R}^m$ with the Lebesgue measure $\mu(d\mathbf{x}) = dx_1 \dots dx_m$.

At any time $t > 0$ the particle, with probability 1, is located in the m-dimensional ball of radius ct

$$B_{ct}^m = \left\{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : x_1^2 + \dots + x_m^2 \leq c^2 t^2\right\}.$$

The distribution $Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$, $\mathbf{x} \in B_{ct}^m$, $t \geq 0$, consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S_{ct}^m = \partial B_{ct}^m = \left\{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \ x_1^2 + \dots + x_m^2 = c^2 t^2\right\}.$$

In this case the particle is located on the sphere S_{ct}^m and the probability of this event is

$$Pr\left\{\mathbf{X}(t) \in S_{ct}^m\right\} = e^{-\lambda t}.$$

If one or more than one Poisson events occur, the particle is located strictly inside the ball B_{ct}^m, and the probability of this event is

$$Pr\left\{\mathbf{X}(t) \in Int\, B_{ct}^m\right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior

$$Int B_{ct}^m = \left\{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : x_1^2 + \dots + x_m^2 < c^2 t^2\right\},$$

and forms its absolutely continuous component.

Therefore there exists the density $p(\mathbf{x}, t) = p(x_1, \dots, x_m, t)$, $\mathbf{x} \in Int\, B_{ct}^m$, $t > 0$, of the absolutely continuous component of the distribution $Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$.

If $N(t) = n$, the displacement of the particle $\mathbf{X}(t)$ at any time $t > 0$ is determined by the coordinates

$$X_k(t) = c \sum_{j=1}^{n+1} (s_j - s_{j-1}) x_k^j, \quad k = 1, \dots, m, \tag{2.1}$$

where x_k^j are the components of the independent m-dimensional random vectors $\mathbf{x}^j = (x_1^j, \dots, x_m^j)$, $j = 1, \dots, n+1$, uniformly distributed on the unit sphere S_1^m; the s_j, $j = 1, \dots, n$ represent the instants at which Poisson events occur, and $s_0 = 0$, $s_{n+1} = t$.

Consider the conditional characteristic functions:

$$H_n(t) = E\left\{e^{i(\boldsymbol{\alpha}, \mathbf{X}(t))} \mid N(t) = n\right\}, \quad n \geq 1, \tag{2.2}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ is the real m-dimensional vector of inversion parameters and $(\boldsymbol{\alpha}, \mathbf{X}(t))$ denotes the inner product of the vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

By substituting (2.1) into (2.2) we have

$$\begin{aligned} H_n(t) &= E\left\{\exp\left(ic\sum_{k=1}^{m}\alpha_k\sum_{j=1}^{n+1}(s_j - s_{j-1})x_k^j\right)\right\} \\ &= E\left\{\exp\left(ic\sum_{j=1}^{n+1}(s_j - s_{j-1})(\boldsymbol{\alpha}, \mathbf{x}^j)\right)\right\}, \quad n \geq 1, \end{aligned}$$

where $(\boldsymbol{\alpha}, \mathbf{x}^j)$ is the inner product of the vectors $\boldsymbol{\alpha}$ and $\mathbf{x}^j$. Computing the expectation in this last equality we obtain

$$H_n(t) = \frac{n!}{t^n}\int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \left\{\prod_{j=1}^{n+1}\left[\frac{1}{\operatorname{mes}(S_1^m)}\int_{S_1^m} e^{ic(\tau_j - \tau_{j-1})(\boldsymbol{\alpha}, \mathbf{x}^j)}\mu(d\mathbf{x}^j)\right]\right\}.$$

The surface integral over the unit sphere S_1^m in this equality can be evaluated by means of Lemma A1 of the Appendix, and is found to be

$$\int_{S_1^m} e^{ic(\tau_j - \tau_{j-1})(\boldsymbol{\alpha}, \mathbf{x}^j)}\mu(d\mathbf{x}^j) = (2\pi)^{m/2}\frac{J_{(m-2)/2}(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)}{(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)^{(m-2)/2}}, \tag{2.3}$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \cdots + \alpha_m^2}$ and $J_{(m-2)/2}(x)$ is the Bessel function of the order $(m-2)/2$ with real argument. Taking into account that

$$\operatorname{mes}(S_1^m) = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})}, \quad m \geq 2,$$

we obtain

$$\begin{aligned} H_n(t) = \frac{n!}{t^n}&\int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \\ &\times \left\{\prod_{j=1}^{n+1}\left[2^{(m-2)/2}\Gamma\left(\frac{m}{2}\right)\frac{J_{(m-2)/2}(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)}{(c(\tau_j - \tau_{j-1})\|\boldsymbol{\alpha}\|)^{(m-2)/2}}\right]\right\}, \quad n \geq 1, \end{aligned} \tag{2.4}$$

(see [5, formula (2) therein], or [12, formula (2.3) therein]).

For the particular cases $m = 2$ (planar motion) and $m = 4$ (four-dimensional motion) the conditional characteristic functions (2.4) were explicitly computed in Kolesnik and Orsingher [7, formula (18) therein] and in Kolesnik [3, formula (10) therein], respectively. However, in the general case for arbitrary $m \geq 2$ the expression on the right-hand side of (2.4) seemingly cannot be explicitly evaluated.

By introducing the function

$$\varphi(t) = 2^{(m-2)/2}\Gamma\left(\frac{m}{2}\right)\frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}}, \quad m \geq 2, \tag{2.5}$$

formula (2.4) can be rewritten as follows

$$H_n(t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \varphi(\tau_j - \tau_{j-1}) \right\}. \tag{2.6}$$

The integral factor in (2.6) can be rewritten in the following form

$$\begin{aligned} \mathcal{I}_n(t) &:= \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \varphi(\tau_j - \tau_{j-1}) \right\} \\ &= \int_0^t d\tau_1 \Bigg\{ \varphi(\tau_1) \int_{\tau_1}^t d\tau_2 \Bigg\{ \varphi(\tau_2 - \tau_1) \\ &\quad \times \int_{\tau_2}^t d\tau_3 \{ \varphi(\tau_3 - \tau_2) \ldots \int_{\tau_{n-2}}^t d\tau_{n-1} \Bigg\{ \varphi(\tau_{n-1} - \tau_{n-2}) \\ &\quad \times \int_{\tau_{n-1}}^t d\tau_n \{ \varphi(\tau_n - \tau_{n-1}) \varphi(t - \tau_n) \} \Bigg\} \cdots \Bigg\} \Bigg\} \Bigg\}. \end{aligned} \tag{2.7}$$

The following theorem states that, for different $n \geq 1$, the functions (2.7) are connected with each other by a convolution-type recurrent relation.

Theorem 1 *For any $n \geq 1$ the following recurrent relation holds*

$$\begin{aligned} \mathcal{I}_n(t) &= \int_0^t \varphi(t - \tau) \mathcal{I}_{n-1}(\tau) d\tau \\ &= \int_0^t \varphi(\tau) \mathcal{I}_{n-1}(t - \tau) d\tau, \quad n \geq 1, \end{aligned} \tag{2.8}$$

where, by definition, $\mathcal{I}_0(x) = \varphi(x)$.

Proof We will prove equality (2.8) by induction. From (2.7), for $n = 1$, we have

$$\begin{aligned} \mathcal{I}_1(t) &= \int_0^t \varphi(\tau) \varphi(t - \tau) d\tau \\ &= \int_0^t \varphi(\tau) \mathcal{I}_0(t - \tau) d\tau \end{aligned} \tag{2.9}$$

and therefore equality (2.8) is valid for $n = 1$.

Suppose that equality (2.8) is valid for all the numbers $k \leq n - 1$, $n \geq 2$. Consider the interior integral (with respect to τ_n) in (2.7). Making in this integral the substitution $\xi = \tau_n - \tau_{n-1}$ we obtain

$$\begin{aligned} \int_{\tau_{n-1}}^t \varphi(\tau_n - \tau_{n-1}) \varphi(t - \tau_n) d\tau_n &= \int_0^{t - \tau_{n-1}} \varphi(\xi) \varphi((t - \tau_{n-1}) - \xi) d\xi \\ &= \mathcal{I}_1(t - \tau_{n-1}) \end{aligned} \tag{2.10}$$

according to (2.9) and induction assumption. The next interior integral (with respect to τ_{n-1}) in (2.7) by means of the substitution $\xi = \tau_{n-1} - \tau_{n-2}$ yields

$$\begin{aligned}\int_{\tau_{n-2}}^{t} &\varphi(\tau_{n-1} - \tau_{n-2})\mathcal{I}_1(t - \tau_{n-1})d\tau_{n-1}\\ &= \int_0^{t-\tau_{n-2}} \varphi(\xi)\mathcal{I}_1((t-\tau_{n-2}) - \xi)d\xi\\ &= \mathcal{I}_2(t - \tau_{n-2})\end{aligned}$$

according to (2.10) and induction assumption.

Continuing this process in the same manner, after the $(n-1)$-th step we obtain

$$\mathcal{I}_n(t) = \int_0^t \varphi(\tau_1)\mathcal{I}_{n-1}(t - \tau_1)d\tau_1,$$

proving (2.8). The theorem is proved. □

Formula (2.8) can be rewritten in the following convolutional form

$$\mathcal{I}_n(t) = \varphi(t) * \mathcal{I}_{n-1}(t) \quad n \geq 1. \tag{2.11}$$

Corollary 1.1 *For any $n \geq 1$ the following relation holds*

$$\mathcal{I}_n(t) = [\varphi(t)]^{*(n+1)}, \quad n \geq 1, \tag{2.12}$$

where the symbol $*(n+1)$ means the $(n+1)$-multiple convolution.

Proof Formula (2.12) automatically follows from (2.11) by means of the chain of equalities

$$\mathcal{I}_n(t) = \varphi(t) * \mathcal{I}_{n-1}(t) = \varphi(t) * \varphi(t) * \mathcal{I}_{n-2}(t) = \cdots = [\varphi(t)]^{*(n+1)}.$$ □

Note that formula (2.12) enables us to immediately write down a *formal* series for the characteristic function $H(t)$ in terms of function $\varphi(t)$, however we postpone doing that till Sect. 4, where a strict justification of such a series, including the proof of its uniform convergence, will be given.

Application of the Laplace transformation

$$\mathcal{L}\left[f(t)\right](s) = \int_0^\infty e^{-st} f(t)dt, \quad \operatorname{Re} s > 0,$$

to the equality (2.12) leads to the following important result.

Corollary 1.2 *For any $n \geq 1$ the Laplace transform of functions (2.7) has the form*

$$\mathcal{L}[\mathcal{I}_n(t)](s) = (\mathcal{L}[\varphi(t)](s))^{n+1}, \quad n \geq 1, \ \operatorname{Re} s > 0. \tag{2.13}$$

Proof The statement immediately follows from the main property of the Laplace transform of convolutions. □

From Theorem 1 it also follows that the conditional characteristic functions are connected with each other by an integral recurrent relation.

Corollary 1.3 *For any $n \geq 1$ the conditional characteristic functions (2.6) satisfy the following recurrent relation*

$$H_n(t) = \frac{n}{t^n} \int_0^t \tau^{n-1} \varphi(t-\tau) H_{n-1}(\tau) d\tau, \quad n \geq 1, \tag{2.14}$$

where $H_0(t) = \varphi(t)$.

Proof Multiplying (2.8) by $(n!/t^n)$, $n \geq 1$, and taking into account (2.6) we obtain

$$\begin{aligned} H_n(t) &= \frac{n}{t^n} \int_0^t \tau^{n-1} \varphi(t-\tau) \left[\frac{(n-1)!}{\tau^{n-1}} \mathcal{I}_{n-1}(\tau) \right] d\tau \\ &= \frac{n}{t^n} \int_0^t \tau^{n-1} \varphi(t-\tau) H_{n-1}(\tau) d\tau. \end{aligned}$$

□

3 Laplace Transforms of Conditional Characteristic Functions

The results of the previous section show that the function $\varphi(t)$ given by (2.5) plays a very important role in our analysis. The reason is that $\varphi(t)$ is exactly the characteristic function (Fourier transform) of the uniform distribution on the surface of the sphere S_{ct}^m of radius ct.

From both the Theorem 1 and its corollaries we see that the conditional characteristic functions $H_n(t)$ and their Laplace transforms, in fact, are expressed in terms of function $\varphi(t)$. Formulae (2.12) and (2.13) show that the possibility of obtaining the explicit form of the conditional characteristic functions (2.6) entirely depends on whether the multiple convolutions of the function $\varphi(t)$ with itself or exact inverse Laplace transforms of its powers can be explicitly evaluated.

In the following theorem we present a general formula for the conditional characteristic functions $H_n(t)$ in terms of inverse Laplace transforms.

Theorem 2 *For any $n \geq 1$ and any $t > 0$ the conditional characteristic functions (2.6) are given by*

$$H_n(t) = \frac{n!}{t^n} \mathcal{L}^{-1} \left[\left(\frac{1}{\sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}} F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2 + (c\|\boldsymbol{\alpha}\|)^2} \right) \right)^{n+1} \right](t), \tag{3.1}$$

where $\mathcal{L}^{-1}$ means the inverse Laplace transform and

$$F(\xi, \eta; \zeta; z) = {}_2F_1(\xi, \eta; \zeta; z) = \sum_{k=0}^{\infty} \frac{(\xi)_k (\eta)_k}{(\zeta)_k} \frac{z^k}{k!}$$

is the Gauss hypergeometric function.

Proof According to Formula 6.621(1) of Gradshteyn and Ryzhik [2] the Laplace transform of the function (2.5) is

$$\mathcal{L}[\varphi(t)](s) = 2^{(m-2)/2} \Gamma\left(\frac{m}{2}\right) \mathcal{L}\left[\frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}} \right](s)$$

$$= \frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}} F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right), \quad \operatorname{Re} s > 0, \tag{3.2}$$

for any $m \geq 2$. Therefore, according to (2.13), we obtain

$$\mathcal{I}_n(t) = \mathcal{L}^{-1}\left[\left(\frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}} F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right)\right)^{n+1}\right](t).$$

Substituting this into (2.6) we obtain (3.1). The theorem is proved. □

Remark 1 Although formula (3.1) is proved for any $n \geq 1$ (and, as we have noted above, this case corresponds to the absolutely continuous component of the distribution of $\mathbf{X}(t)$), one can easily show that formula (3.1) is also valid for $n = 0$ (and this case corresponds to the singular component of the distribution).

Really, for $n = 0$, formula (3.1) formally yields

$$\begin{aligned} H_0(t) &= \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}} F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right)\right](t) \\ &= 2^{(m-2)/2}\Gamma\left(\frac{m}{2}\right) \frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}}, \end{aligned} \tag{3.3}$$

where we have used a formula of the inverse Laplace transform for the Gauss hypergeometric function (see [1, Table 5.19, formula 6]), (see also (3.2)).

On the other hand, by applying Lemma A1 of the Appendix it is easy to show that

$$\begin{aligned} H_0(t) &= E\left\{e^{i(\boldsymbol{\alpha},\mathbf{X}(t))} | N(t) = 0\right\} \\ &= \frac{\Gamma(\frac{m}{2})}{2\pi^{m/2}(ct)^{m-1}} \int_{S_{ct}^m} e^{i(\boldsymbol{\alpha},\mathbf{x})} \mu(d\mathbf{x}) \\ &= \frac{\Gamma(\frac{m}{2})}{2\pi^{m/2}} \int_{S_1^m} e^{ict(\boldsymbol{\alpha},\mathbf{x})} \mu(d\mathbf{x}) \\ &= 2^{(m-2)/2}\Gamma\left(\frac{m}{2}\right) \frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}}, \end{aligned}$$

and this coincides with (3.3). As we have noted above, formula (3.3) represents the characteristic function (Fourier transform) of the uniform distribution on the surface of the sphere S_{ct}^m of radius ct.

Remark 2 By applying Formula 9.131(1) of Gradshteyn and Ryzhik [2] the hypergeometric function can be rewritten as

$$\frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}} F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right) = \frac{1}{s} F\left(\frac{1}{2}, 1; \frac{m}{2}; -\frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2}\right),$$

and therefore formula (3.1) has the following alternative form

$$H_n(t) = \frac{n!}{t^n}\mathcal{L}^{-1}\left[\left(\frac{1}{s} F\left(\frac{1}{2}, 1; \frac{m}{2}; -\frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2}\right)\right)^{n+1}\right](t).$$

Corollary 2.1 *The characteristic function of the random vector* $\mathbf{X}(t)$, $t \geq 0$, *has the following series representation*

$$H(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n \mathcal{L}^{-1}\left[\left(\frac{1}{s} F\left(\frac{1}{2}, 1; \frac{m}{2}; -\frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2}\right)\right)^{n+1}\right](t).$$

Proof The statement immediately follows from (3.1) and Remarks 1 and 2. □

Let us demonstrate how our technique developed above works in some important particular cases.

3.1 Two-dimensional case

In the planar case $m = 2$ and therefore the function (2.5) has the form

$$\varphi(t) = J_0(ct\|\boldsymbol{\alpha}\|), \qquad \|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2}.$$

Then, according to (2.13), we have

$$\mathcal{L}[\mathcal{I}_n(t)](s) = (\mathcal{L}[J_0(ct\|\boldsymbol{\alpha}\|)](s))^{n+1}.$$

Taking into account (see, for instance [9, Table 8.4-1, formula 55] that

$$\mathcal{L}\,[J_0(ct\|\boldsymbol{\alpha}\|)]\,(s) = \frac{1}{\sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}}$$

we obtain

$$\mathcal{L}\,[\mathcal{I}_n(t)]\,(s) = \frac{1}{(s^2 + (c\|\boldsymbol{\alpha}\|)^2)^{(n+1)/2}}.$$

According to Korn and Korn [9, Table 8.4-1, formula 57] the inverse Laplace transformation of this function yields

$$\begin{aligned} \mathcal{I}_n(t) &= \mathcal{L}^{-1}\left[\frac{1}{(s^2 + (c\|\boldsymbol{\alpha}\|)^2)^{(n+1)/2}}\right](t) \\ &= \frac{\sqrt{\pi}}{\Gamma(\frac{n+1}{2})}\left(\frac{t}{2c\|\boldsymbol{\alpha}\|}\right)^{n/2} J_{n/2}(ct\|\boldsymbol{\alpha}\|). \end{aligned}$$

Then the conditional characteristic functions (2.6) have the form

$$H_n(t) = \frac{n!\sqrt{\pi}}{2^{n/2}\Gamma(\frac{n+1}{2})} \frac{J_{n/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n/2}}, \quad n \geq 1. \tag{3.4}$$

By duplication formula for gamma-function we have

$$n! = \Gamma\left(2 \cdot \frac{n+1}{2}\right) = \frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2} + 1\right).$$

Substituting this into (3.4) we finally obtain

$$H_n(t) = 2^{n/2}\Gamma\left(\frac{n}{2} + 1\right)\frac{J_{n/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n/2}}, \quad n \geq 1, \tag{3.5}$$

and this coincides with formula (18) of Kolesnik and Orsingher [7] obtained by other more complicated way.

We can come to the same result by using Theorem 2. From formula (3.1) and taking into account Korn and Korn [9, Table 8.4-1, formula 57] we immediately get

$$\begin{aligned} H_n(t) &= \frac{n!}{t^n}\mathcal{L}^{-1}\left[\frac{1}{(s^2+(c\|\boldsymbol{\alpha}\|)^2)^{(n+1)/2}}\right](t) \\ &= \frac{n!\sqrt{\pi}}{2^{n/2}\Gamma(\frac{n+1}{2})}\frac{J_{n/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n/2}}, \end{aligned}$$

and thus we again obtain (3.4).

The inverse Fourier transformation of the functions (3.5) with respect to $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ leads to the following conditional distributions (see [7, Theorem 1])

$$\begin{aligned} &Pr\{\mathbf{X}(t)\in d\mathbf{x}\mid N(t)=n\} = \frac{n}{2\pi(ct)^2}\left(1-\frac{\|\mathbf{x}\|^2}{c^2t^2}\right)^{(n-2)/2}\mu(d\mathbf{x}), \quad n\ge 1, \\ &\mathbf{X}(t)=(X_1(t),X_2(t)), \qquad \mathbf{x}=(x_1,x_2)\in Int\, B_{ct}^2, \qquad \|\mathbf{x}\|^2=x_1^2+x_2^2, \\ &\mu(d\mathbf{x})=dx_1dx_2, \end{aligned}$$

where B_{ct}^2 is the planar disc of radius ct.

3.2 Four-dimensional case

In this case $m=4$ and therefore the function (2.5) takes the form

$$\varphi(t)=2\frac{J_1(ct\|\boldsymbol{\alpha}\|)}{ct\|\boldsymbol{\alpha}\|}, \qquad \|\boldsymbol{\alpha}\|=\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2}.$$

According to (2.13), we have

$$\mathcal{L}[\mathcal{I}_n(t)](s)=2^{n+1}\left(\mathcal{L}\left[\frac{J_1(ct\|\boldsymbol{\alpha}\|)}{ct\|\boldsymbol{\alpha}\|}\right](s)\right)^{n+1}.$$

Taking into account (see, for instance [9, Table 8.4-1, formula 58]) that

$$\mathcal{L}\left[\frac{J_1(ct\|\boldsymbol{\alpha}\|)}{ct\|\boldsymbol{\alpha}\|}\right](s)=\frac{1}{(c\|\boldsymbol{\alpha}\|)^2}\left(\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}-s\right)$$

we obtain

$$\mathcal{L}[\mathcal{I}_n(t)](s)=\frac{2^{n+1}}{(c\|\boldsymbol{\alpha}\|)^{2n+2}}\left(\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}-s\right)^{n+1}.$$

According to the same formula, the inverse Laplace transformation of this function yields

$$\begin{aligned} \mathcal{I}_n(t) &= \frac{2^{n+1}}{(c\|\boldsymbol{\alpha}\|)^{2n+2}}\mathcal{L}^{-1}\left[\left(\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}-s\right)^{n+1}\right](t) \\ &= \frac{2^{n+1}(n+1)}{(c\|\boldsymbol{\alpha}\|)^{n+1}}\frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{t}. \end{aligned}$$

Then the conditional characteristic functions (2.6) have the form

$$H_n(t) = \frac{n!}{t^n} \frac{2^{n+1}(n+1)}{(c\|\boldsymbol{\alpha}\|)^{n+1}} \frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{t}$$
$$= 2^{n+1}(n+1)! \frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n+1}}, \quad n \geq 1, \tag{3.6}$$

and this coincides with formula (10) of Kolesnik [3] obtained by other method.

The same result can be obtained directly from Theorem 2. Really, formula (3.1) yields

$$H_n(t) = \frac{n!}{t^n} \mathcal{L}^{-1}\left[\left(\frac{1}{\sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}} F\left(\frac{1}{2}, 1; 2; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2 + (c\|\boldsymbol{\alpha}\|)^2} \right) \right)^{n+1} \right](t).$$

By applying Formula 9.121(24) of Gradshteyn and Ryzhik [2] we can easily show that

$$F\left(\frac{1}{2}, 1; 2; \frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2 + (c\|\boldsymbol{\alpha}\|)^2} \right) = \frac{2\sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}}{s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2}}, \quad \operatorname{Re} s > 0, \tag{3.7}$$

and, therefore,

$$H_n(t) = \frac{2^{n+1} n!}{t^n} \mathcal{L}^{-1}\left[\left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2} \right)^{-(n+1)} \right](t).$$

According to Bateman and Erdelyi [1, Table 5.3, formula 43 or Table 5.4, formula 21]

$$\mathcal{L}^{-1}\left[\left(s + \sqrt{s^2 + (c\|\boldsymbol{\alpha}\|)^2} \right)^{-(n+1)} \right](t) = (c\|\boldsymbol{\alpha}\|)^{-(n+1)}(n+1) \frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{t}.$$

Hence,

$$H_n(t) = \frac{2^{n+1} n!}{t^n} (c\|\boldsymbol{\alpha}\|)^{-(n+1)}(n+1) \frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{t}$$
$$= 2^{n+1}(n+1)! \frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n+1}},$$

and thus we again obtain formula (3.6).

The inverse Fourier transformation of the functions (3.6) with respect to $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ leads to the following conditional distributions (see [3, Theorem 1])

$$Pr\{\mathbf{X}(t) \in d\mathbf{x} \mid N(t) = n\} = \frac{n(n+1)}{\pi^2 (ct)^4} \left(1 - \frac{\|\mathbf{x}\|^2}{c^2 t^2} \right)^{n-1} \mu(d\mathbf{x}), \quad n \geq 1,$$
$$\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t)), \qquad \mathbf{x} = (x_1, x_2, x_3, x_4) \in Int\, B_{ct}^4,$$
$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \qquad \mu(d\mathbf{x}) = dx_1 dx_2 dx_3 dx_4,$$

where B_{ct}^4 is the four-dimensional ball of radius ct.

3.3 Three-dimensional case

Our analysis loses its simplicity if $m=3$. In this case the function (2.5) takes the form

$$\varphi(t)=\sqrt{2}\frac{\sqrt{\pi}}{2}\frac{J_{1/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{1/2}}=\frac{\sin(ct\|\boldsymbol{\alpha}\|)}{ct\|\boldsymbol{\alpha}\|},\quad \|\boldsymbol{\alpha}\|=\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}.$$

According to (2.13), we have

$$\mathcal{L}[\mathcal{I}_n(t)](s)=(c\|\boldsymbol{\alpha}\|)^{-(n+1)}\left(\mathcal{L}\left[\frac{\sin(ct\|\boldsymbol{\alpha}\|)}{t}\right](s)\right)^{n+1}.$$

Taking into account (see, for instance [9, Table 8.4-1, formula 107]) that

$$\mathcal{L}\left[\frac{\sin(ct\|\boldsymbol{\alpha}\|)}{t}\right](s)=\operatorname{arctg}\frac{c\|\boldsymbol{\alpha}\|}{s}$$

we obtain

$$\mathcal{L}[\mathcal{I}_n(t)](s)=(c\|\boldsymbol{\alpha}\|)^{-(n+1)}\left(\operatorname{arctg}\frac{c\|\boldsymbol{\alpha}\|}{s}\right)^{n+1}.$$

Thus,

$$\mathcal{I}_n(t)=(c\|\boldsymbol{\alpha}\|)^{-(n+1)}\mathcal{L}^{-1}\left[\left(\operatorname{arctg}\frac{c\|\boldsymbol{\alpha}\|}{s}\right)^{n+1}\right](t).$$

Therefore, according to (2.6), the conditional characteristic functions $H_n(t)$ have the following form

$$H_n(t)=\frac{n!}{t^n}(c\|\boldsymbol{\alpha}\|)^{-(n+1)}\mathcal{L}^{-1}\left[\left(\operatorname{arctg}\frac{c\|\boldsymbol{\alpha}\|}{s}\right)^{n+1}\right](t),\quad n\geq 1. \tag{3.8}$$

We now show how formula (3.8) can be derived directly from Theorem 2. According to (3.1) we have

$$H_n(t)=\frac{n!}{t^n}\mathcal{L}^{-1}\left[\left(\frac{1}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right)\right)^{n+1}\right](t). \tag{3.9}$$

By applying formula 9.121(26) of Gradshteyn and Ryzhik [2] it's easy to see that

$$\begin{aligned}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{(c\|\boldsymbol{\alpha}\|)^2}{s^2+(c\|\boldsymbol{\alpha}\|)^2}\right)&=\frac{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}{c\|\boldsymbol{\alpha}\|}\arcsin\left(\frac{c\|\boldsymbol{\alpha}\|}{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}\right)\\&=\frac{\sqrt{s^2+(c\|\boldsymbol{\alpha}\|)^2}}{c\|\boldsymbol{\alpha}\|}\operatorname{arctg}\frac{c\|\boldsymbol{\alpha}\|}{s}.\end{aligned} \tag{3.10}$$

Substituting this into (3.9) we obtain (3.8).

The inverse Laplace transform on the right-hand side of (3.8), apparently, cannot be explicitly computed for arbitrary $n\geq 1$. However, for the important particular case $n=1$ (corresponding to the single change of direction) expression (3.8) can be evaluated explicitly.

From (3.8) and taking into account formula (7.4) of Lemma A2 of the Appendix (see below), we obtain

$$\begin{aligned}
H_1(t) &= \frac{1}{t}(c\|\boldsymbol{\alpha}\|)^{-2}\mathcal{L}^{-1}\left[\left(\operatorname{arctg}\frac{c\|\boldsymbol{\alpha}\|}{s}\right)^2\right](t) \\
&= \frac{1}{(ct\|\boldsymbol{\alpha}\|)^2}\big[\sin(ct\|\boldsymbol{\alpha}\|)\mathrm{Si}(2ct\|\boldsymbol{\alpha}\|) + \cos(ct\|\boldsymbol{\alpha}\|)\mathrm{Ci}(2ct\|\boldsymbol{\alpha}\|)\big], \qquad (3.11)
\end{aligned}$$

where the functions $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are the incomplete integral sine and cosine, respectively, given by (7.3) (see below).

The expression (3.11) coincides with formula (6) of Kolesnik [4]. Its inverse Fourier transformation with respect to $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ leads to the following conditional distribution (see [4])

$$\begin{aligned}
&Pr\{\mathbf{X}(t) \in d\mathbf{x} \mid N(t) = 1\} = \frac{1}{4\pi(ct)^2\|\mathbf{x}\|}\ln\left(\frac{ct+\|\mathbf{x}\|}{ct-\|\mathbf{x}\|}\right)\mu(d\mathbf{x}), \\
&\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t)), \qquad \mathbf{x} = (x_1, x_2, x_3) \in Int\, B_{ct}^3, \\
&\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \mu(d\mathbf{x}) = dx_1 dx_2 dx_3, \qquad (3.12)
\end{aligned}$$

where B_{ct}^3 is the three-dimensional ball of radius ct. Formula (3.12) represents the discontinuous term of the distribution of the random vector $\mathbf{X}(t)$, $t > 0$. Note that (3.12) is similar, for $c = 1$, to the second term of formulae (1.3) and (4.21) of Stadje [16].

4 Integral Equation for Characteristic Function

According to (2.6), the characteristic function of $\mathbf{X}(t)$, $t \geq 0$, is given by the formal series

$$\begin{aligned}
H(t) &= E\left\{e^{i(\boldsymbol{\alpha}, \mathbf{X}(t))}\right\} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}H_n(t) \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n \mathcal{I}_n(t). \qquad (4.1)
\end{aligned}$$

We now prove that the series on the right-hand side of (4.1) converges uniformly (with respect to $\|\boldsymbol{\alpha}\|$) for any $t \geq 0$. The easily checked inequality

$$\left|\frac{J_\nu(x)}{x^\nu}\right| \leq \frac{1}{2^\nu\Gamma(\nu+1)}, \qquad \nu \geq 0,$$

implies

$$|\varphi(t)| = 2^{(m-2)/2}\Gamma\left(\frac{m}{2}\right)\left|\frac{J_{(m-2)/2}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{(m-2)/2}}\right| \leq 1 \qquad (4.2)$$

for any $m \geq 2$. By induction it is easy to show that for any $n \geq 1$ the following equality holds

$$\int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n = \frac{t^n}{n!}, \qquad n \geq 1. \qquad (4.3)$$

Therefore, taking into account (4.2), (4.3) and remembering that $\mathcal{I}_0(t)=\varphi(t)$, we obtain

$$\begin{aligned}\left|\sum_{n=0}^{\infty}\lambda^n\mathcal{I}_n(t)\right| &\le |\varphi(t)|+\sum_{n=1}^{\infty}\lambda^n\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\cdots\int_{\tau_{n-1}}^t d\tau_n\left\{\prod_{j=1}^{n+1}|\varphi(\tau_j-\tau_{j-1})|\right\}\\ &\le 1+\sum_{n=1}^{\infty}\lambda^n\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\cdots\int_{\tau_{n-1}}^t d\tau_n\\ &=1+\sum_{n=1}^{\infty}\frac{\lambda^n t^n}{n!}\\ &=e^{\lambda t}<\infty\end{aligned}$$

for any $t\ge 0$ and all $\|\boldsymbol{\alpha}\|$. Hence, the series in (4.1) converges uniformly with respect to $\|\boldsymbol{\alpha}\|$ for any $t\ge 0$ and, therefore, it uniquely determines some smooth function which, being multiplied by $e^{-\lambda t}$, produces the characteristic function $H(t)$ of the random vector $\mathbf{X}(t)$, $t\ge 0$.

In the following theorem we present an integral equation for the function $H(t)$ and its explicit form in terms of function $\varphi(t)$.

Theorem 3 *The characteristic function $H(t)$, $t\ge 0$, satisfies the following convolution-type Volterra integral equation of second kind with the continuous kernel $e^{-\lambda t}\varphi(t)$:*

$$H(t)=e^{-\lambda t}\varphi(t)+\lambda\int_0^t e^{-\lambda(t-\tau)}\varphi(t-\tau)H(\tau)d\tau,\quad t\ge 0. \tag{4.4}$$

In the class of continuous functions the integral equation (4.4) *has the unique solution given by*

$$H(t)=e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n[\varphi(t)]^{*(n+1)}. \tag{4.5}$$

Proof According to Theorem 1 and (4.1), we have

$$\begin{aligned}H(t)&=e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n\mathcal{I}_n(t)\\ &=e^{-\lambda t}\left\{\varphi(t)+\sum_{n=1}^{\infty}\lambda^n\int_0^t\varphi(t-\tau)\mathcal{I}_{n-1}(\tau)d\tau\right\}\\ &\qquad\text{(uniform convergence of the series)}\\ &=e^{-\lambda t}\left\{\varphi(t)+\int_0^t\varphi(t-\tau)\left(\sum_{n=1}^{\infty}\lambda^n\mathcal{I}_{n-1}(\tau)\right)d\tau\right\}\\ &=e^{-\lambda t}\left\{\varphi(t)+\int_0^t\varphi(t-\tau)\left(\sum_{n=0}^{\infty}\lambda^{n+1}\mathcal{I}_n(\tau)\right)d\tau\right\}\\ &=e^{-\lambda t}\left\{\varphi(t)+\lambda\int_0^t\varphi(t-\tau)\left(\sum_{n=0}^{\infty}\lambda^n\mathcal{I}_n(\tau)\right)d\tau\right\}\end{aligned}$$

$$= e^{-\lambda t} \left\{ \varphi(t) + \lambda \int_0^t \varphi(t-\tau) e^{\lambda \tau} H(\tau) d\tau \right\}$$

$$= e^{-\lambda t} \varphi(t) + \lambda \int_0^t e^{-\lambda(t-\tau)} \varphi(t-\tau) H(\tau) d\tau$$

proving (4.4).

The integral equation (4.4) can be rewritten in the following convolutional form

$$H(t) = e^{-\lambda t}\varphi(t) + \lambda[(e^{-\lambda t}\varphi(t)) * H(t)], \quad t \geq 0. \tag{4.6}$$

By direct substituting (4.5) into (4.6) we can easily check that the function (4.5) is really the solution to the convolutional equation (4.6). Its uniqueness follows from the well-known fact that any Volterra integral equation of second kind with continuous kernel has the unique solution in the class of continuous functions for any λ and arbitrary continuous free term. Thus, the theorem is completely proved. □

We should note that, although formula (4.5) gives the general form of the characteristic function $H(t)$, the multiple convolutions of the function $\varphi(t)$ with itself seemingly cannot be explicitly evaluated in arbitrary dimension.

Remark 3 From (4.6) we immediately obtain the general formula for the Laplace transform of the characteristic function $H(t)$:

$$\mathcal{L}[H(t)](s) = \frac{\mathcal{L}[\varphi(t)](s+\lambda)}{1 - \lambda \mathcal{L}[\varphi(t)](s+\lambda)}, \quad \operatorname{Re} s > 0. \tag{4.7}$$

By using (3.2) we can rewrite (4.7) in the explicit form

$$\mathcal{L}[H(t)](s) = \frac{F(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2})}{\sqrt{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2} - \lambda F(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2})}. \tag{4.8}$$

In particular, in the planar case ($m = 2$) formula (4.8) takes the form

$$\mathcal{L}[H(t)](s) = \frac{1}{\sqrt{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2} - \lambda},$$

and this coincides with formula (12) of Masoliver *et al.* [11].

In the three-dimensional case ($m = 3$), in view of (3.10), formula (4.8) yields

$$\mathcal{L}[H(t)](s) = \frac{\operatorname{arctg}(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda})}{c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda})}.$$

This exactly coincides with formula (45) of Masoliver *et al.* [11] and, for $c = 1$, with formulae (1.6) and (5.8) of Stadje [16].

Finally, in the four-dimensional case ($m = 4$), in view of (3.7), formula (4.8) becomes

$$\mathcal{L}[H(t)](s) = \frac{2}{s + \sqrt{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2} - \lambda}.$$

Remark 4 Similarly to Theorem 3 one can show that the characteristic function of the absolutely continuous component of the distribution of $\mathbf{X}(t)$, $t>0$, defined by

$$\widetilde{H}(t)=e^{-\lambda t}\sum_{n=1}^{\infty}\lambda^n\mathcal{I}_n(t),$$

satisfies the Volterra integral equation

$$\widetilde{H}(t)=\lambda e^{-\lambda t}\mathcal{I}_1(t)+\lambda\int_0^t e^{-\lambda(t-\tau)}\varphi(t-\tau)\widetilde{H}(\tau)d\tau,\quad t>0, \tag{4.9}$$

where $\mathcal{I}_1(t)$ is given by (2.9).

Equation (4.9) can be rewritten in the convolutional form as follows

$$\widetilde{H}(t)=\lambda e^{-\lambda t}[\varphi(t)*\varphi(t)]+\lambda\left[\left(e^{-\lambda t}\varphi(t)\right)*\widetilde{H}(t)\right],\quad t>0. \tag{4.10}$$

Therefore, the Laplace transform of the function $\widetilde{H}(t)$ has the form

$$\mathcal{L}[\widetilde{H}(t)](s)=\frac{\lambda\{\mathcal{L}[\varphi(t)]\}^2(s+\lambda)}{1-\lambda\mathcal{L}[\varphi(t)](s+\lambda)},\quad \operatorname{Re}s>0. \tag{4.11}$$

One can easily check that the function

$$\widetilde{H}(t)=e^{-\lambda t}\sum_{n=1}^{\infty}\lambda^n[\varphi(t)]^{*(n+1)}. \tag{4.12}$$

is the unique continuous solution to (4.10).

From (4.9) it follows as well that the function

$$\overline{H}(t)=e^{\lambda t}\widetilde{H}(t)=\sum_{n=1}^{\infty}\lambda^n\mathcal{I}_n(t),$$

satisfies the more simple Volterra integral equation

$$\overline{H}(t)=\lambda\mathcal{I}_1(t)+\lambda\int_0^t\varphi(t-\tau)\overline{H}(\tau)d\tau,\quad t>0, \tag{4.13}$$

or, in the convolutional form,

$$\overline{H}(t)=\lambda[\varphi(t)*\varphi(t)]+\lambda[\varphi(t)*\overline{H}(t)],\quad t>0. \tag{4.14}$$

The Laplace transform of the function $\overline{H}(t)$ is given by

$$\mathcal{L}[\overline{H}(t)](s)=\frac{\lambda\{\mathcal{L}[\varphi(t)](s)\}^2}{1-\lambda\mathcal{L}[\varphi(t)](s)},\quad \operatorname{Re}s>0, \tag{4.15}$$

and the function

$$\overline{H}(t)=\sum_{n=1}^{\infty}\lambda^n[\varphi(t)]^{*(n+1)}. \tag{4.16}$$

is the unique continuous solution to (4.14).

Remark 5 Here we give the complete solution of the long-standing problem of finding the initial conditions for the partial differential equations governing random evolutions. Although such governing equations are known for the one and two-dimensional evolutions only, nevertheless Theorem 3 enables us to write down initial conditions for *any* dimension without knowing any special differential relations.

The equality

$$\lim_{x\to 0}\frac{J_\nu(x)}{x^\nu}=\frac{1}{2^\nu\Gamma(\nu+1)},\qquad \nu\geq 0,$$

implies

$$\varphi(t)|_{t=0}=1. \tag{4.17}$$

Then we obtain from (4.4)

$$H(t)|_{t=0}=1. \tag{4.18}$$

Therefore, the transition density $f(\mathbf{x},t)$, $\mathbf{x}\in\mathbb{R}^m$, $t\geq 0$, of the process $\mathbf{X}(t)$ satisfies the first initial condition

$$f(\mathbf{x},t)|_{t=0}=\delta(\mathbf{x}), \tag{4.19}$$

where $\delta(\mathbf{x})$ is the m-dimensional Dirac delta-function. The condition (4.19) expresses the obvious fact that at the initial moment $t=0$ the distribution is entirely concentrated in the origin.

The difficulty of finding the second initial condition was pointed out by many authors. Theorem 3 enables us to easily obtain the second initial condition without any additional differential relations. Really, differentiating (4.4) with respect to t we have

$$\frac{\partial H(t)}{\partial t}=-\lambda e^{-\lambda t}\varphi(t)+e^{-\lambda t}\frac{\partial\varphi(t)}{\partial t}+\lambda H(t)+\lambda\int_0^t\frac{\partial}{\partial t}\left[e^{-\lambda(t-\tau)}\varphi(t-\tau)\right]H(\tau)d\tau.$$

Then, by taking into account (4.17), (4.18) and the easily checked equality

$$\left.\frac{\partial\varphi(t)}{\partial t}\right|_{t=0}=0, \tag{4.20}$$

we immediately obtain

$$\left.\frac{\partial H(t)}{\partial t}\right|_{t=0}=0. \tag{4.21}$$

Therefore the second initial condition in *any* dimension has the form

$$\left.\frac{\partial f(\mathbf{x},t)}{\partial t}\right|_{t=0}=0. \tag{4.22}$$

If necessary, we are able to continue this procedure and find the next initial conditions for the characteristic function $H(t)$. For instance, by differentiating (4.4) twice with respect to t, we have

$$\frac{\partial^2 H(t)}{\partial t^2} = \lambda^2 e^{-\lambda t}\varphi(t) - 2\lambda e^{-\lambda t}\frac{\partial\varphi(t)}{\partial t} + e^{-\lambda t}\frac{\partial^2\varphi(t)}{\partial t^2} + \lambda\frac{\partial H(t)}{\partial t} - \lambda^2 H(t) + \lambda\int_0^t \frac{\partial^2}{\partial t^2}\big[e^{-\lambda(t-\tau)}\varphi(t-\tau)\big]H(\tau)d\tau.$$

From this, by taking into account (4.17), (4.18), (4.20), (4.21) and the easily checked equality

$$\left.\frac{\partial^2\varphi(t)}{\partial t^2}\right|_{t=0} = -\frac{(c\|\boldsymbol{\alpha}\|)^2}{m},$$

we obtain

$$\left.\frac{\partial^2 H(t)}{\partial t^2}\right|_{t=0} = -\frac{(c\|\boldsymbol{\alpha}\|)^2}{m}.$$

It's interesting to note that this formula explicitly depends on the dimension m.

5 Limit Theorem

One of the most remarkable features of the transport processes in lower dimensions is their weak convergence to the Brownian motions as both the particle's speed c and the intensity of switches λ tend to infinity in such a way that the following Kac condition holds

$$c\to\infty, \qquad \lambda\to\infty, \quad \frac{c^2}{\lambda}\to\rho, \quad \rho>0. \tag{5.1}$$

Our technique enables us to extend this very important result for the isotropic transport process in the Euclidean space $\mathbb{R}^m$ of arbitrary dimension $m\ge 2$.

Theorem 4 *Under the Kac condition (5.1) the transition density of the isotropic transport process $\mathbf{X}(t)$ converges to the transition density of the homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = 2\rho/m$, that is,*

$$\lim_{\substack{c,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} p(\mathbf{x},t) = \left(\frac{m}{4\rho\pi t}\right)^{m/2}\exp\left(-\frac{m}{4\rho t}\|\mathbf{x}\|^2\right), \quad m\ge 2, \tag{5.2}$$

where $\|\mathbf{x}\|^2 = x_1^2+\cdots+x_m^2$.

Proof Under the Kac condition (5.1) we have

$$\lim_{\substack{c,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2} = 0$$

and therefore

$$\lim_{\substack{c,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} F\left(\frac{1}{2},\frac{m-2}{2};\frac{m}{2};\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right) = 1.$$

Then by passing to the limit in (4.8) under the Kac condition (5.1) we obtain

$$
\begin{aligned}
&\lim_{\substack{c,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \mathcal{L}[H(t)](s)\\
&= \lim_{\substack{c,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \left[\sqrt{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2} - \lambda\, F\left(\frac{1}{2}, \frac{m-2}{2}; \frac{m}{2}; \frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)\right]^{-1}\\
&= \lim_{\substack{c,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \Bigg[(s+\lambda)\sqrt{1+\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2}\\
&\quad - \lambda\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{m-2}{2})_k}{(\frac{m}{2})_k}\frac{1}{k!}\left(\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)^k\Bigg]^{-1}
\end{aligned} \tag{5.3}
$$

From the Kac condition (5.1) it follows that for sufficiently large c and λ the inequality

$$\left|\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right| < 1$$

holds for any s and $\|\boldsymbol{\alpha}\|$. Therefore, the radical in (5.3) can be represented in the form of the absolutely converging series

$$\sqrt{1+\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2} = 1+\frac{1}{2}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2 - \frac{1\cdot 1}{2\cdot 4}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^4 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^6 - \cdots.$$

Substituting this into (5.3) we can rewrite it as follows

$$
\begin{aligned}
&\lim_{\substack{c,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \mathcal{L}[H(t)](s)\\
&= \lim_{\substack{c,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \Bigg[(s+\lambda)\left(1+\frac{1}{2}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^2 - \frac{1\cdot 1}{2\cdot 4}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)^4 + \cdots\right)\\
&\quad - \lambda\left(1 + \frac{\left(\frac{1}{2}\right)_1\left(\frac{m-2}{2}\right)_1}{\left(\frac{m}{2}\right)_1}\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right.\\
&\quad \left. + \frac{1}{2!}\frac{(\frac{1}{2})_2(\frac{m-2}{2})_2}{(\frac{m}{2})_2}\left(\frac{(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\right)^2 + \ldots\right)\Bigg]^{-1}\\
&= \lim_{\substack{c,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \Bigg[s+\lambda+\frac{1}{2}\frac{(c\|\boldsymbol{\alpha}\|)^2}{s+\lambda} - \frac{1\cdot 1}{2\cdot 4}\frac{(c\|\boldsymbol{\alpha}\|)^4}{(s+\lambda)^3} + \cdots\\
&\quad - \lambda - \frac{(\frac{1}{2})_1(\frac{m-2}{2})_1}{(\frac{m}{2})_1}\frac{\lambda(c\|\boldsymbol{\alpha}\|)^2}{(s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2}\\
&\quad - \frac{1}{2!}\frac{(\frac{1}{2})_2(\frac{m-2}{2})_2}{(\frac{m}{2})_2}\frac{\lambda(c\|\boldsymbol{\alpha}\|)^4}{((s+\lambda)^2+(c\|\boldsymbol{\alpha}\|)^2)^2} - \cdots\Bigg]^{-1}
\end{aligned}
$$

$$= \lim_{\substack{c,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \Bigg[s + \frac{1}{2} \frac{\frac{c^2}{\lambda}\|\boldsymbol{\alpha}\|^2}{\frac{s}{\lambda}+1} - \frac{1\cdot 1}{2\cdot 4} \frac{\frac{c^4}{\lambda^3}\|\boldsymbol{\alpha}\|^4}{(\frac{s}{\lambda}+1)^3} + \cdots$$

$$- \frac{(\frac{1}{2})_1(\frac{m-2}{2})_1}{(\frac{m}{2})_1} \frac{\frac{c^2}{\lambda}\|\boldsymbol{\alpha}\|^2}{(\frac{s}{\lambda}+1)^2 + \frac{c^2}{\lambda^2}\|\boldsymbol{\alpha}\|^2}$$

$$- \frac{1}{2!} \frac{(\frac{1}{2})_2(\frac{m-2}{2})_2}{(\frac{m}{2})_2} \frac{\frac{c^4}{\lambda^3}\|\boldsymbol{\alpha}\|^4}{((\frac{s}{\lambda}+1)^2 + \frac{c^2}{\lambda^2}\|\boldsymbol{\alpha}\|^2)^2} - \cdots \Bigg]^{-1}$$

Taking into account that, under the Kac condition (5.1), $(c^n/\lambda^{n-1}) \to 0$ for any $n \geq 3$ (see also [8, formula (4.4)]), we obtain

$$\lim_{\substack{c,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \mathcal{L}[H(t)](s) = \left[s + \frac{1}{2}\rho\|\boldsymbol{\alpha}\|^2 - \frac{(\frac{1}{2})_1(\frac{m-2}{2})_1}{(\frac{m}{2})_1}\rho\|\boldsymbol{\alpha}\|^2 \right]^{-1}.$$

It's easy to check that

$$\frac{(\frac{1}{2})_1(\frac{m-2}{2})_1}{(\frac{m}{2})_1} = \frac{m-2}{2m}, \qquad m \geq 2.$$

Thus, we finally obtain

$$\lim_{\substack{c,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \mathcal{L}[H(t)](s) = \left(s + \frac{\rho\|\boldsymbol{\alpha}\|^2}{m} \right)^{-1}. \tag{5.4}$$

Inverse Laplace transformation of the function (5.4) yields

$$\mathcal{L}^{-1}\left[\left(s + \frac{\rho\|\boldsymbol{\alpha}\|^2}{m} \right)^{-1} \right](t) = \exp\left(-\frac{\rho\|\boldsymbol{\alpha}\|^2}{m} t \right), \tag{5.5}$$

where we have used Bateman and Erdelyi [1, Table 5.2, formula 1]. The function on the right-hand side of (5.5) is the characteristic function of the m-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = 2\rho/m$.

By applying the Hankel inversion formula we can easily show that the inverse Fourier transformation $\mathcal{F}^{-1}$ of the function on the right-hand side of (5.5) yields

$$\mathbf{w}(\mathbf{x}, t) = \mathcal{F}^{-1}\big[e^{-(\rho\|\boldsymbol{\alpha}\|^2 t)/m}\big] = \left(\frac{m}{4\rho\pi t} \right)^{m/2} \exp\left(-\frac{m\|\mathbf{x}\|^2}{4\rho t} \right), \tag{5.6}$$

and this coincides with the function on the right-hand side of (5.2). The theorem is thus completely proved. □

The function (5.6) is exactly the transition density of the m-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = 2\rho/m$. This entirely accords with some previous results concerning the limiting behaviour of isotropic transport processes (see, for comparison, Papanicolaou [13, p. 353, the Theorem], Pinsky [14, Proposition 4.8]).

It's easy to see that if $m = 2$ and $\rho = 1$, the density (5.6) turns into the transition density of the two-dimensional standard Brownian motion (see [7, p. 1181]). For $m = 4$ and $\rho = 1$ the density (5.6) turns into the transition density of the four-dimensional Brownian motion obtained in Kolesnik [3, formula (21)]. Note also that if we set $\rho = m/2$, the limiting process becomes the m-dimensional standard homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = 1$.

We can easily check that the density (5.6) is the fundamental solution to the m-dimensional heat equation

$$\frac{\partial \mathbf{w}(\mathbf{x}, t)}{\partial t} = \frac{\rho}{m} \Delta \mathbf{w}(\mathbf{x}, t), \tag{5.7}$$

where Δ denotes the m-dimensional Laplacian. For $\rho = 1$ the differential operator on the right-hand side of (5.7) exactly coincides with the generator obtained by Pinsky [14, Proposition 4.8].

6 Non-Symmetrical Random Motions

The basic feature of the motion studied above was the uniform choice of both the initial and each new direction at every Poissonian instant. This key property provided the absolute spatial symmetry of the process $\mathbf{X}(t)$. The symmetrical structure of $\mathbf{X}(t)$ is clearly seen from the form of its characteristic functions $H_n(t)$ and $H(t)$ which contain the inversion multi-parameter $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ as the symmetric functional $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \cdots + \alpha_m^2}$. Surprisingly, the majority of the results obtained above keep their validity also for the non-symmetrical motions.

Suppose that both the initial and every new direction are taken on according to some arbitrary distribution on the unit sphere S_1^m. Let $\chi(\mathbf{x})$, $\mathbf{x} \in S_1^m$, denote the density of this distribution, assumed to exist. Let $\mathbf{Z}(t) = (Z_1(t), \dots, Z_m(t))$ be the particle's position at an arbitrary instant $t > 0$. Consider the conditional characteristic functions (Fourier transform)

$$G_n(t) = E\left\{ e^{i(\boldsymbol{\alpha}, \mathbf{Z}(t))} \,|\, N(t) = n \right\}, \quad n \geq 1, \tag{6.1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ is the real m-dimensional vector of inversion parameters.

Similarly to the symmetrical case, the functions (6.1) can be written as follows

$$G_n(t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \left[\int_{S_1^m} e^{ic(\tau_j - \tau_{j-1})(\boldsymbol{\alpha}, \mathbf{x}^j)} \, \chi(\mathbf{x}^j) \, \mu(d\mathbf{x}^j) \right] \right\}. \tag{6.2}$$

By introducing the function

$$\psi(t) = \int_{S_1^m} e^{ict(\boldsymbol{\alpha}, \mathbf{x})} \chi(\mathbf{x}) \mu(d\mathbf{x}) \tag{6.3}$$

we can rewrite (6.2) in the following form

$$G_n(t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \psi(\tau_j - \tau_{j-1}) \right\}, \quad n \geq 1. \tag{6.4}$$

Note that the function $\psi(t)$ given by (6.3) represents the characteristic function (Fourier transform) of the density $\chi(\mathbf{x})$ on the surface of the sphere S_{ct}^m of radius ct.

Denote the integral factor in (6.4) as follows

$$\mathcal{J}_n(t) = \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \psi(\tau_j - \tau_{j-1}) \right\}, \quad n \geq 1. \tag{6.5}$$

The following non-symmetrical counterparts of Theorem 1 and its corollaries take place.

Theorem 5 *For any $n \geq 1$ the following recurrent relation holds*

$$\begin{aligned} \mathcal{J}_n(t) &= \int_0^t \psi(t-\tau) \mathcal{J}_{n-1}(\tau) d\tau \\ &= \int_0^t \psi(\tau) \mathcal{J}_{n-1}(t-\tau) d\tau, \quad n \geq 1, \end{aligned} \tag{6.6}$$

where, by definition, $\mathcal{J}_0(x) = \psi(x)$.

Formula (6.6) can be rewritten in the following convolutional form

$$\mathcal{J}_n(t) = \psi(t) * \mathcal{J}_{n-1}(t) \quad n \geq 1. \tag{6.7}$$

Corollary 5.1 *For any $n \geq 1$ the following relation holds*

$$\mathcal{J}_n(t) = \left[\psi(t)\right]^{*(n+1)}, \quad n \geq 1, \tag{6.8}$$

where the symbol $*(n+1)$ means the $(n+1)$-multiple convolution.

Corollary 5.2 *For any $n \geq 1$ the Laplace transform of functions (6.5) has the form*

$$\mathcal{L}[\mathcal{J}_n(t)](s) = (\mathcal{L}[\psi(t)](s))^{n+1}, \quad n \geq 1. \tag{6.9}$$

Corollary 5.3 *For any $n \geq 1$ the conditional characteristic functions (6.4) satisfy the following recurrent relation*

$$G_n(t) = \frac{n}{t^n} \int_0^t \tau^{n-1} \psi(t-\tau) G_{n-1}(\tau) d\tau, \quad n \geq 1, \tag{6.10}$$

where $G_0(t) = \psi(t)$.

The proofs of Theorem 5 and Corollaries 5.1, 5.2, 5.3 are the simple recompilations of the proofs of Theorem 1 and Corollaries 1.1, 1.2, 1.3, respectively, in which the function $\varphi(t)$ is everywhere replaced by the function $\psi(t)$ and, therefore, omitted.

The characteristic function of $\mathbf{Z}(t)$ given by the uniformly converging series

$$G(t) = E\left\{e^{i(\boldsymbol{\alpha}, \mathbf{Z}(t))}\right\} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} G_n(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n \mathcal{J}_n(t). \tag{6.11}$$

satisfies a Volterra integral equation. This result is given by the following theorem.

Theorem 6 *The characteristic function $G(t)$, $t \geq 0$, satisfies the following convolution-type Volterra integral equation of second kind with the continuous kernel $e^{-\lambda t}\psi(t)$:*

$$G(t) = e^{-\lambda t}\psi(t) + \lambda \int_0^t e^{-\lambda(t-\tau)}\psi(t-\tau)G(\tau)d\tau, \quad t \geq 0. \tag{6.12}$$

In the class of continuous functions the integral equation (6.12) has the unique solution given by

$$G(t) = e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n \left[\psi(t)\right]^{*(n+1)}. \tag{6.13}$$

The proof is similar to the proof of Theorem 3, and therefore omitted.

The integral equation (6.12) can be rewritten in the following convolutional form

$$G(t) = e^{-\lambda t}\psi(t) + \lambda[(e^{-\lambda t}\psi(t)) * G(t)], \quad t \geq 0. \tag{6.14}$$

From this we obtain the general formula for the Laplace transform of the characteristic function $G(t)$:

$$\mathcal{L}[G(t)](s) = \frac{\mathcal{L}[\psi(t)](s+\lambda)}{1 - \lambda\mathcal{L}[\psi(t)](s+\lambda)}, \quad \operatorname{Re} s > 0. \tag{6.15}$$

One can show that the characteristic function of the absolutely continuous component of the distribution of $\mathbf{Z}(t)$, $t > 0$, defined by

$$\widetilde{G}(t) = e^{-\lambda t}\sum_{n=1}^{\infty}\lambda^n \mathcal{J}_n(t),$$

satisfies the Volterra integral equation

$$\widetilde{G}(t) = \lambda e^{-\lambda t}\mathcal{J}_1(t) + \lambda \int_0^t e^{-\lambda(t-\tau)}\psi(t-\tau)\widetilde{G}(\tau)d\tau, \quad t > 0, \tag{6.16}$$

where

$$\mathcal{J}_1(t) = \psi(t) * \psi(t) = \int_0^t \psi(\tau)\psi(t-\tau)d\tau.$$

Equation (6.16) can be rewritten in the convolutional form as follows

$$\widetilde{G}(t) = \lambda e^{-\lambda t}[\psi(t) * \psi(t)] + \lambda\left[\left(e^{-\lambda t}\psi(t)\right) * \widetilde{G}(t)\right], \quad t > 0. \tag{6.17}$$

Therefore, the Laplace transform of the function $\widetilde{G}(t)$ has the form

$$\mathcal{L}\left[\widetilde{G}(t)\right](s) = \frac{\lambda\{\mathcal{L}[\psi(t)]\}^2(s+\lambda)}{1 - \lambda\mathcal{L}[\psi(t)](s+\lambda)}, \quad \operatorname{Re} s > 0. \tag{6.18}$$

One can easily check that the function

$$\widetilde{G}(t) = e^{-\lambda t}\sum_{n=1}^{\infty}\lambda^n[\psi(t)]^{*(n+1)}. \tag{6.19}$$

is the unique continuous solution to (6.17).

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Appendix

In this Appendix we prove two auxiliary lemmas which have been used in our analysis. The first one concerns the integration of a complex exponential over the surface of the unit sphere S_1^m.

Lemma A1 *For any dimension $m \geq 2$ and for arbitrary real constant C the following relation holds*

$$\int_{S_1^m} e^{iC(\boldsymbol{\alpha},\mathbf{x})}\mu(d\mathbf{x}) = (2\pi)^{m/2}\frac{J_{(m-2)/2}(C\|\boldsymbol{\alpha}\|)}{(C\|\boldsymbol{\alpha}\|)^{(m-2)/2}}. \tag{7.1}$$

Proof According to Formula 4.644 of Gradshteyn and Ryzhik [2], for any $m \geq 2$, we have

$$\begin{aligned}\int_{S_1^m} e^{iC(\boldsymbol{\alpha},\mathbf{x})}\mu(d\mathbf{x}) &= \int\cdots\int_{x_1^2+\cdots+x_m^2=1} e^{iC(\alpha_1 x_1+\cdots+\alpha_m x_m)}dx_1\dots dx_m \\ &= \frac{2\pi^{(m-1)/2}}{\Gamma(\frac{m-1}{2})}\int_0^{\pi} e^{iC\|\boldsymbol{\alpha}\|\cos\theta}(\sin\theta)^{m-2}d\theta \\ &= (2\pi)^{m/2}\frac{J_{(m-2)/2}(C\|\boldsymbol{\alpha}\|)}{(C\|\boldsymbol{\alpha}\|)^{(m-2)/2}},\end{aligned}$$

where in the last step we have used the well-known integral representation of the Bessel function (see, for instance [2, Formula 8.411(7)]). The lemma is proved. □

Note that for $m = 2$ formula (7.1) yields the well-known integral representation

$$\begin{aligned}\int_{S_1^2} e^{iC(\boldsymbol{\alpha},\mathbf{x})}\mu(d\mathbf{x}) &= \iint_{x_1^2+x_2^2=1} e^{iC(\alpha_1 x_1+\alpha_2 x_2)}dx_1 dx_2 \\ &= \int_0^{2\pi} e^{iC(\alpha_1\cos\theta+\alpha_2\sin\theta)}d\theta \\ &= 2\pi J_0(C\|\boldsymbol{\alpha}\|).\end{aligned}$$

For $m = 3$ equality (7.1) transforms into the well-known formula

$$\begin{aligned}\int_{S_1^3} e^{iC(\boldsymbol{\alpha},\mathbf{x})}\mu(d\mathbf{x}) &= \iiint_{x_1^2+x_2^2+x_3^2=1} e^{iC(\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3)}dx_1 dx_2 dx_3 \\ &= (2\pi)^{3/2}\frac{J_{1/2}(C\|\boldsymbol{\alpha}\|)}{(C\|\boldsymbol{\alpha}\|)^{1/2}} \\ &= 4\pi\frac{\sin(C\|\boldsymbol{\alpha}\|)}{C\|\boldsymbol{\alpha}\|}.\end{aligned}$$

The second auxiliary lemma concerns the Laplace transform and inverse Laplace transform which have been used in Sect. 3.3. This lemma is of a separate interest because, to the best of our knowledge, there are no formulae similar to (7.2) or (7.4) (see below) in the handbooks on integral transforms, including those in the reference list.

Lemma A2 *For arbitrary real constant k the following formula holds*

$$\mathcal{L}\left[\frac{\sin(kt)}{t}\mathrm{Si}(2kt)+\frac{\cos(kt)}{t}\mathrm{Ci}(2kt)\right](s)=\left(\mathrm{arctg}\frac{k}{s}\right)^2, \quad \mathrm{Re}\, s>0, \tag{7.2}$$

where $\mathcal{L}$ means the Laplace transform and the functions $\mathrm{Si}(x)$ *and* $\mathrm{Ci}(x)$ *are the incomplete integral sine and cosine*, *respectively, given by*

$$\mathrm{Si}(x)=\int_0^x \frac{\sin\xi}{\xi}d\xi, \qquad \mathrm{Ci}(x)=\int_0^x \frac{\cos\xi-1}{\xi}d\xi. \tag{7.3}$$

The inverse Laplace transformation of (7.2) *yields*

$$\mathcal{L}^{-1}\left[\left(\mathrm{arctg}\frac{k}{s}\right)^2\right](t)=\frac{1}{t}[\sin(kt)\mathrm{Si}(2kt)+\cos(kt)\mathrm{Ci}(2kt)]. \tag{7.4}$$

Proof We have

$$\begin{aligned}
\frac{\sin(kt)}{t} * \frac{\sin(kt)}{t} &= \int_0^t \frac{\sin(k\tau)}{\tau}\frac{\sin(k(t-\tau))}{t-\tau}d\tau \\
&= \frac{1}{t}\int_0^t \sin(k\tau)\sin(k(t-\tau))\left(\frac{1}{\tau}+\frac{1}{t-\tau}\right)d\tau \\
&= \frac{2}{t}\int_0^t \frac{\sin(k\tau)\sin(k(t-\tau))}{\tau}d\tau \\
&= \frac{2}{t}\int_0^t \frac{\sin(k\tau)}{\tau}[\sin(kt)\cos(k\tau)-\sin(k\tau)\cos(kt)]d\tau \\
&= \frac{\sin(kt)}{t}\int_0^t \frac{2\sin(k\tau)\cos(k\tau)}{\tau}d\tau - \frac{\cos(kt)}{t}\int_0^t \frac{2\sin^2(k\tau)}{\tau}d\tau \\
&= \frac{\sin(kt)}{t}\int_0^t \frac{\sin(2k\tau)}{\tau}d\tau - \frac{\cos(kt)}{t}\int_0^t \frac{1-\cos(2k\tau)}{\tau}d\tau \\
&= \frac{\sin(kt)}{t}\mathrm{Si}(2kt)+\frac{\cos(kt)}{t}\mathrm{Ci}(2kt).
\end{aligned}$$

Applying now the Laplace transformation to both sides of this equality and taking into account that

$$\mathcal{L}\left[\frac{\sin(kt)}{t}\right](s)=\mathrm{arctg}\frac{k}{s}$$

(see, for instance [9, Table 8.4-1, formula 107]) we obtain (7.2). The lemma is proved. □

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